Representability and the Scope of the Mentaculus

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Draft of: February 28, 2026

Abstract

Statistical Mechanical Imperialism (SMI) holds that the laws and probabilities of the special sciences can, in principle, be recovered from microphysics together with a small set of statistical-mechanical postulates known as the Mentaculus. Central to this claim is the idea that special-scientific explanations can be vindicated by assigning probabilities to special-science propositions via a probability measure over initial microstates. In this paper, I argue that this strategy relies on a substantive but largely unacknowledged assumption: that special-science propositions correspond to subsets of phase space that are measurable by the probability measure defined by the Mentaculus. I call this the Probabilistic Representability Requirement (PRR). I show that the PRR neither follows from the core commitments of the Mentaculus nor enjoys independent justification. Moreover, without the PRR, the Mentaculus fails to assign probabilities to special-science propositions, and its explanatory ambitions cannot be sustained.

1 Introduction: Imperialism and Its Formal Commitments

Statistical Mechanical Imperialism[1] (SMI) is a philosophical position asserting that all the laws and probabilities in the special sciences can, in principle, be reduced to, or fully explained by, microphysics together with a set of statistical-mechanical postulates known as *the Mentaculus* (Albert, 2012; Loewer, 2020a). The framework presupposes a division between *microstates* and *macrostates*. Microstates constitute complete physical descriptions of the universe, determining all physical facts. Macrostates, by contrast, are coarse-grainings of microstates: they provide higher-level descriptions that fix the properties and locations of middle-sized and large objects. For present purposes, macrostates may therefore be identified with certain sets of microstates.

Given deterministic microphysical laws, which the Mentaculus assumes and I grant, each complete micro-history of the universe is fixed by its initial microstate, specified by the particle positions,

[1] This term was coined, somewhat provocatively, by Weslake (2014).

momenta, and intrinsic properties (e.g., charge). Thus, each proposition can be identified—modulo the microphysics—with the set of initial microstates of the universe that realize it. This identification makes it natural to assign probabilities to propositions by defining a measure over the space of initial microstates. Within this setting, the Mentaculus is specified by two postulates:

The Past Hypothesis (PH). The initial state of the universe is one of very low entropy.

The Probability Postulate (PROB). The probability of the universe being in a certain macroscopic state at a time t is given by the uniform (Lebesgue) measure over the set of initial microstates that evolve into it by time t, conditional on PH.

This can be specified diagramatically, as illustrated in Figure 1, by considering the set $M_{\Diamond}$ of physically possible initial microstates of the universe, and the set $M(0) \subseteq M_{\Diamond}$ of initial microstates that realize PH. The probability postulate then assigns a *probability* to each macro-level proposition P (represented as a set of initial microstates) by measuring $M(0) \cap P$ with the (normalized) uniform measure on $M(0)$.

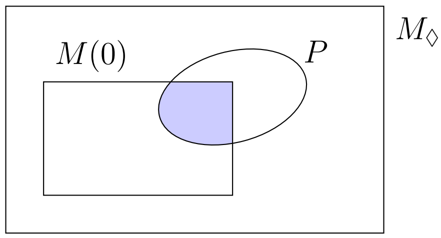

Figure 1: The objective chance of P according to the Mentaculus.

Proponents of SMI argue that PH and PROB should be accepted as laws over and above the fundamental dynamical laws, on grounds of simplicity and explanatory power. The uniform measure is taken to be the simplest empirically adequate distribution over the microstates compatible with PH, and PROB underwrites the standard statistical-mechanical explanation of thermodynamic behavior. Moreover, SMI proponents claim that the same framework suffices, in principle, to explain laws and regularities across the special sciences. The general strategy can be summarized as follows:

The Recipe (Albert, 2012, p. 30). Suppose a special science explains some fact E by appealing to certain features F of the world, and suppose we wish to recover this explanation within the Mentaculus framework. Then, we can proceed as follows:

(S1) Start with the probability distribution we just defined, call it **Pr**, and measure the conditional probability $\mathbf{Pr}(E|F)$ of the proposition E given F.

(S2) If this conditional probability is high, then the special-scientific explanation of E is recovered from microphysics plus the Mentaculus.

(S3) If not, either the microphysical theory, the special-scientific explanation, or both must be revised.

To make things more concrete, let's consider a familiar special-science explanation in the literature. Consider the explanation of why the introduction of large amounts of paper money into the German economy in the 1920s led to the disappearance of gold coins from circulation (Loewer, 2009, pp. 226–227). At the level of economics, this phenomenon is explained by appeal to Gresham's Law: *ceteris paribus*, bad money drives out good money.

In terms of the Recipe, E would be the set of initial microstates of the universe which evolve into a state in which gold coins dissapeared from circulation in 1920s Germany, and F would be the set of those which evolve into a state in which large quantities of paper money were introduced into the German economy in the 1920s and the *ceteris paribus* condition of Gresham's Law is satisfied.

Thus, (S1) requires us to evaluate the conditional probability $\mathbf{Pr}(E|F)$, which, following PROB, is given by the ratio of the uniform measure of $E \cap F \cap M(0)$ over that of $F \cap M(0)$. In (S2), we then evaluate whether it is high enough. If it is, then the Mentaculus has successfully "recovered" this special-scientific explanation. If not, then we get to (S3): either the microphysical theory, the special-scientific explanation (including Gresham's Law), or both must be revised.

Of course, the Recipe's success rests on a number of substantive assumptions. First, it requires that the sets E and F of initial microstates of the universe are well-defined. This amounts to a form of supervenience physicalism, according to which special-scientific facts supervene on microphysical facts:

Micro-physical supervenience (MPS). Any two worlds that are physically possible relative to the microphysics and agree on the microphysical facts also agree on the special-scientific facts.

For the purposes of this paper, I will grant MPS. A further assumption, however, has received far less scrutiny, despite being tacitly relied upon. Step (S1) presupposes that the *probabilities* of the sets corresponding to E and F are themselves well-defined by PROB—that is, that these sets fall within the domain of this probability measure, presupposing the following:

Probabilistic representability requirement (PRR). For any adequate special-scientific explanation of the fact E by means of the features F, the sets $E \cap F \cap M(0)$ and $F \cap M(0)$ are measurable by the probability measure determined by PROB.

The goal of the present paper is to argue that, on the usual formulation of the Mentaculus, this assumption is not justified, and that, absent further justification, we have no reason to accept it.

But before we get to that, it is helpful to introduce some context and to locate this claim within the broader debate over the Mentaculus.

2 Existing challenges and the current challenge

Existing objections to SMI include doubts that the Mentaculus explains the *nomic* status of special science laws (Fodor, 1974), doubts about the nomic status of principles like PH and PROB (e.g. Dorst 2024; Chen 2023), and concerns about the justification for relying on the Mentaculus' predictions beyond thermodynamics (Weslake, 2014). For present purposes, I bracket these issues and grant that PH and PROB are laws, that the Recipe delivers genuine recovers special-scientific laws explanations when successful, and that defeasible optimism about its verdicts outside thermodynamics is warranted.

The challenge developed here is orthogonal: whether the Mentaculus can assign probabilities to special-science propositions at all. More precisely, I argue that step (S1) of the Recipe requires the *Probabilistic Representability Requirement* (PRR), as formulated above. I then show that PRR neither follows from core assumptions of the Mentaculus, nor enjoys independent justification. Moreover, without PRR, the Mentaculus fails to assign probabilities to special-science propositions, and its explanatory ambitions cannot be sustained.

3 The Probabilistic Representability Requirement

The Probabilistic Representability Requirement (PRR) is the assumption that the Mentaculus assigns probabilities to all propositions required for special-scientific explanations—that is, that it assigns probabilities to all propositions expressible in the languages of the special sciences that feature in their explanations. PRR functions as a bridge principle connecting the microphysical probability measure defined by PROB with propositions expressed in special-science vocabularies. It is therefore indispensable to SMI's claim that the Mentaculus provides a "complete scientific theory of the universe" (Albert, 2012, p. 21): without PRR, the probabilities required to implement (S1) of the Recipe are not guaranteed to exist. I will argue that, upon closer inspection, there is little reason to accept PRR without introducing additional assumptions.

To see how PRR enters the Recipe, note that (S1) requires the conditional probability $\mathbf{Pr}(E|F)$ of a special-scientific fact E obtaining given that the world has certain explanatorily relevant features

F. Classically, conditional probabilities are defined as follows:

$$\mathbf{Pr}(E|F) = \frac{\mathbf{Pr}(E \cap F)}{\mathbf{Pr}(F)}.$$

Accordingly, for $\mathbf{Pr}(E|F)$ to be well-defined, both $\mathbf{Pr}(F)$ and $\mathbf{Pr}(E \cap F)$ must themselves be well-defined.

In the Mentaculus framework, however, **Pr** is defined via PROB as the normalized Lebesgue measure of the subsets of initial microstates of $M(0)$. But standard measure theory does not guarantee that the Lebesgue measure assigns a value to every subset of $M(0)$. Indeed, assuming the Axiom of Choice, there exist subsets of $M(0)$ that are non-measurable—and therefore, subsets to which **Pr** assigns no probability at all (Halmos, 1950, pp. 67–72). If either $F \cap M(0)$ or $E \cap F \cap M(0)$ are non-measurable, then the conditional probability $\mathbf{Pr}(E|F)$ is undefined, and (S1) cannot be carried out.

Since SMI aims to recover *all* adequate special-scientific explanations by means of the Recipe, it is therefore committed to the assumption that every special-scientific proposition that figures in those explanations corresponds to a subset of $M(0)$ that is measurable by the probability measure determined by PROB. This is precisely the Probabilistic Representability Requirement.

Crucially, nothing in the standard formulation of the Mentaculus licenses this assumption. Supervenience assumptions such as MPS guarantee only that the sets of microstates corresponding to special-scientific facts are well-defined, not that they are measurable. Nor do PH or PROB themselves provide any further resources here, since PH isn't probabilistic, and PROB already presupposes the Lebesgue measure in defining **Pr**. PRR is therefore an additional and substantive assumption—one that requires independent justification if the explanatory ambitions of SMI are to be sustained.

4 Possible replies

There are a number of possible replies to the claim that the PRR lacks justification. In this section, I consider, to my mind, the most salient replies on behalf of SMI and argue that none succeeds without introducing substantial assumptions.

4.1 Denying the Axiom of Choice

One way to resist the representability challenge is to deny the mathematical result on which it relies. Since the existence of non-measurable sets depends on the Axiom of Choice (AC), one might argue that AC should be rejected, or at least suspended, in the present context. If AC is denied, then the existence of non-measurable subsets of $M(0)$ can no longer be established in the usual way, and the present argument would not go through.

This response, however, is not a serious option for defenders of SMI. AC is standard and overwhelmingly accepted in contemporary mathematical practice. Results proved using AC are ordinarily regarded as fully legitimate—e.g., proving a Millenium Prize Problem using AC would entitle the solver to the $1 million prize—, and many central theorems across analysis, topology, and algebra rely on it. Moreover, AC is equivalent to a number of claims that are independently intuitive, such as the assertion that the Cartesian product of any collection of nonempty sets is itself nonempty.

More importantly, rejecting AC solely in order to block the existence of non-measurable sets would be *ad hoc*. The explanatory ambitions of SMI do not come with any independent motivation for revising set theory, nor is it plausible that the success of a philosophical account of scientific explanation should take precedence over independently established mathematical principles. If forced to choose between the Axiom of Choice and the unrestricted success of SMI, the former is overwhelmingly better supported.

4.2 Non-measurable sets are irrelevant or negligible

A more concessive response grants the Axiom of Choice and the existence of non-measurable sets, but maintains that such sets are nevertheless irrelevant to the present discussion. The underlying thought is that, even if non-measurable subsets of $M(0)$ exist, they are too exceptional or marginal to plausibly correspond to propositions of the special sciences.

It is important to distinguish two different ways of developing this idea. On one reading, the claim is that non-measurable sets are irrelevant because they are rare—"few and far between" within the space of subsets of $M(0)$. On another, the claim is that non-measurable sets are irrelevant because they are not constructible or definable in any explicit way.

Understood as a claim about rarity, the objection fails on straightforward cardinality grounds. Given AC and plausible assumptions about $M(0)$'s cardinality, non-measurable subsets of $M(0)$ are not rare: they are as numerous, and varied[2], as all subsets of $M(0)$. A basic cardinality analysis is given in the Appendix.

But what if we understand this claim as a claim about the unconstructibility or undefinability of non-measurable sets? This calls for an altogether different response.

[2]The sense in which they are "varied", and why this is relevant, is given in the Appendix.

4.3 Non-measurable sets are gerrymandered, or "unnatural"

A third response grants that non-measurable sets exist and are plentiful, but maintains that they are nevertheless irrelevant because they are excessively gerrymandered, or insufficiently natural to correspond to propositions of the special sciences. According to this line of thought, even if non-measurable subsets of $M(0)$ are abundant in the abstract, the special sciences traffic only in natural kinds and well-behaved properties, and there is no reason to expect such properties to have non-measurable microphysical correlates.

There are several points to be made in response. First, appeals to naturalness introduce substantive constraints that go beyond the standard formulation of the Mentaculus. Neither PH nor PROB, nor the Recipe itself, imposes any restriction on which subsets of $M(0)$ may represent special-scientific propositions beyond those required for supervenience. If the defender of SMI appeals to naturalness to restrict the propositions expressible by special-sciences, this constitutes an additional assumption—one that requires independent justification. The present challenge therefore stands: without additional substantial assumptions, PRR remains unmotivated.

Second, it is important to be clear about the sense in which non-measurable sets are alleged to be gerrymandered. Non-measurability does not imply gerrymandering or "unnaturalness" *simpliciter*, but only gerrymandering *relative to the microphysics*. Yet this is precisely the sense in which special-science predicates are widely acknowledged to be gerrymandered by participants on all sides of the reductionism debate.

This point is familiar from the Fodor–Loewer debate. Here, Fodor (1974) appeals to the disjunctive and heterogeneous microphysical bases of special-science predicates in order to argue against reduction, and Loewer (2009) *explicitly concedes* this heterogeneity while denying that it blocks in-principle metaphysical reduction. Once this concession is in place, however, appeals to microphysical gerrymandering lose their force as an exclusion criterion for candidates sets of initial microstates corresponding to special-science propositions, since it is *already granted* that these sets are gerrymandered from the microphysical point of view.

Third, even if one were to appeal to a more refined notion of naturalness—one intended to mediate between special-science predicates and the microphysics, perhaps via law-based or functional criteria—the connection to measurability remains unestablished. Recent proposals along these lines (e.g., Gómez Sánchez, 2023) attempt to articulate this notion, but no argument has been given to show that such constraints guarantee something like the PRR. At most, these approaches gesture at a possible strategy for supplementing the Mentaculus, not at a consequence of its existing commitments.

4.4 Epistemic decidability

A more focused version of the naturalness reply appeals not merely to microphysical gerrymandering, but to epistemic decidability. The suggestion is that non-measurable sets are "unnatural" in a crucial sense: membership in such sets does not appear to be the sort of physical fact that could be established by human observation, whether unaided or aided by arbitrarily advanced technology. Special-scientific propositions, by contrast, are taken to concern properties that can in principle be detected or measured. Call this the *Observability Constraint* (OC). The thought is that non-measurable sets fail this constraint, and therefore cannot correspond to the propositions of the special sciences. There are two difficulties with this proposal.

The first difficulty concerns OC itself. It is implausible that epistemic decidability is a necessary condition for participation in special-scientific explanations. Our ordinary and scientific vocabularies contain many predicates whose application we cannot reliably determine in arbitrary cases, and whose extension is highly theory-laden. It would be a strong—and controversial—verificationist thesis to hold that only predicates whose extension can in principle be observationally settled count as legitimate. The present dialectic does not warrant such a commitment.

Second, and more importantly, explanations of how we come to know whether a particular ordinary or special-scientific proposition holds plausibly belong to the special sciences themselves—to linguistics, cognitive science, biology, etc. If so, insisting that the epistemic decidability of a proposition requires its corresponding set of initial microstates to be measurable effectively presupposes that these sciences already reduce in the appropriate way to the Mentaculus. But that is precisely what is at issue.

None of this shows that no suitable epistemic or representational constraint could ultimately secure something like the PRR. It may well be that a principled account of naturalness, projectibility, or cognitive accessibility can be articulated in a way that guarantees measurability. The present point is more modest: such a guarantee is not supplied by the standard formulation of the Mentaculus itself, and appeals to observability do not bridge the gap without additional substantial assumptions.

4.5 Defeasible optimism from thermodynamic success

A final response concedes that the PRR has not been formally justified, but maintains that this is not troubling. The empirical success of statistical mechanics in thermodynamics, it is argued, provides defeasible reason to expect that the Mentaculus can be extended to other special-scientific domains.

This reply, however, conflates two distinct questions. One concerns the *reliability* of the Mentaculus when it applies: do its probabilistic predictions tend to be correct? The other concerns its *scope*: in which domains does it deliver probabilistic verdicts at all? The present challenge bears on the latter question.

In thermodynamics, the Mentaculus yields concrete predictions because macroscopic properties such as entropy and temperature are explicitly represented in microphysical and statistical-mechanical terms.[3] Outside thermodynamics, by contrast, we lack clear cases in which special-scientific explananda have been successfully represented as measurable subsets of the space of initial microstates, let alone cases in which the relevant probabilities have been computed.

The problem, then, is not merely that the Mentaculus has been insufficiently tested beyond thermodynamics. It is that we have little reason to think it delivers predictions there at all. And this is precisely what one would expect if the PRR does not hold. Appeals to past success therefore do not secure the extension of the framework to domains in which probabilistic representability has yet to be established.

5 Conclusion

The claim that the Mentaculus assigns probabilities to the propositions required for special-scientific explanations is not secured by its standard formulation. Even granting its empirical success in thermodynamics and its core reductionist commitments, the framework relies on an additional Probabilistic Representability Requirement: that special-science propositions correspond to measurable subsets of phase space. I have argued that this requirement neither follows from the Mentaculus nor is justified by common replies appealing to mathematical practice, naturalness, or defeasible optimism from past success. Without further representational constraints, the imperialist ambitions of the Mentaculus remain incomplete. The broader moral is that debates over the scope of unifying scientific frameworks must attend not only to where they succeed empirically, but also to where they are able to make predictions at all.

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[3]See, e.g., Reif (2009, ch. 3).

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Appendix

For the purpose of cardinality considerations, the Lebesgue measure on $M(0)$ can be considered as equivalent to the one on the unit interval $[0, 1]$, since both have the cardinality of the continuum $\mathfrak{c}$, and a sufficiently similar topology. So, let λ be the Lebesgue measure on the unit interval $[0, 1]$.

Theorem 1. *If the Axiom of Choice is true, there are* $2^{\mathfrak{c}}$ *non-*λ*-measurable subsets of* $[0, 1]$.

Proof. The Axiom of Choice implies the existence of a non-measurable subset of $[0, 1]$ (Halmos, 1950, p. 67). Let V be a non-measurable subset of $[0, 1]$ and let C be the Cantor set. This set has cardinality $\mathfrak{c}$ and λ-measure 0. Since its cardinality is $\mathfrak{c}$, there are $2^{\mathfrak{c}}$ subsets S of C, and because λ is a complete measure, each of these subsets has λ-measure 0. Now, consider the mapping

$$f : \mathcal{P}(C) \to \mathcal{P}([0, 1]) \mid f(S) = V \triangle S,$$

where the symmetric difference $V \triangle S$ is defined as

$$V \triangle S := (V \setminus S) \cup (S \setminus V) = \{x \in [0, 1] : x \in V \text{ or } x \in S \text{ but not both}\}.$$

It can be easily shown that f is an injective mapping, which means that there are at least $2^{\mathfrak{c}}$ sets of the form $V \triangle S$, where $S \subseteq C$. Now, since S is measurable and measurability is closed under symmetric difference, $V \triangle S$ must also be non-λ-measurable, since otherwise $V = (V \triangle S) \triangle S$ would be measurable. Thus, the set $\{V \triangle S : S \subseteq C\}$ contains only non-λ-measurable sets and has a cardinality of $2^{\mathfrak{c}}$. □

One might object that Theorem 1 is something of a cheap trick. After all, starting from a single non-measurable set V, we generated $2^{\mathfrak{c}}$ further non-measurable sets by taking symmetric differences $V \triangle S$ with subsets S of measure zero. But each of these sets differs from V only by a λ-null set (i.e. S). A natural response would therefore be to identify sets that differ only by a null set, and to ask whether non-measurable sets remain abundant once this identification is made. The following theorem shows that they do. In fact, when sets are so identified, non-measurable equivalence classes vastly outnumber measurable ones. In this sense, non-measurable sets are not merely numerous, but significantly more varied than measurable sets.

Theorem 2. *Define* $A, B \subseteq [0, 1]$ *to be* λ-equivalent *(written* $A \sim_\lambda B$*) iff* $A \triangle B$ *is* λ*-measurable and* $\lambda(A \triangle B) = 0$. *For each* $A \subseteq [0, 1]$, *let* $[A]_\lambda$ *denote its* λ*-equivalence class, and define* $[0, 1]_\lambda = \{[A]_\lambda : A \subseteq [0, 1]\}$. *Then each equivalence class* $[A]_\lambda$ *contains either only* λ*-measurable sets or only non-*λ*-measurable sets. Moreover, assuming the Axiom of Choice, there are at most* $\mathfrak{c}$ *measurable equivalence classes and at least* $2^{\mathfrak{c}}$ *non-measurable ones.*

Proof. First, we show that no equivalence class mixes measurable and non-measurable sets. Let $A, B \subseteq [0,1]$ be such that $A \sim_\lambda B$. Then $A \triangle B$ is λ-measurable and $\lambda(A \triangle B) = 0$. If A is λ-measurable, then $B = (A \triangle B) \triangle A$ is λ-measurable since measurability is closed under symmetric difference. For the same reason, if A is non-λ-measurable, then B cannot be λ-measurable, since otherwise $A = (A \triangle B) \triangle B$ would be λ-measurable, a contradiction. Hence every equivalence class $[A]_\lambda$ contains either only measurable or only non-measurable sets.

Next, we bound the number of measurable equivalence classes. By definition, the Lebesgue measure is the completion of Borel measure, so every λ-measurable set differs from some Borel set by a λ-null measurable set. Therefore every measurable equivalence class contains a Borel representative. Since the family of Borel subsets of $[0,1]$ has cardinality $\mathfrak{c}$, there are at most $\mathfrak{c}$ measurable equivalence classes.

Finally, we show that there are at least $2^{\mathfrak{c}}$ non-measurable equivalence classes. Let $\mathcal{Z}$ be the family of null Borel subsets of [0,1]. Since there are only $\mathfrak{c}$ many Borel sets, we have $|\mathcal{Z}| = \mathfrak{c}$, and hence $|\mathcal{Z} \times \mathfrak{c}| = \mathfrak{c}$. Use the well-ordering theorem to fix an enumeration $\{(N_\alpha, \kappa_\alpha) : \alpha < \mathfrak{c}\}$ of $\mathcal{Z} \times \mathfrak{c}$. Now, for each $\alpha < \mathfrak{c}$, we have $|[0,1] \setminus N_\alpha| = \mathfrak{c}$, so by transfinite induction, we can recursively choose a distinct point $x_\alpha \in [0,1] \setminus N_\alpha$ for each $\alpha < \mathfrak{c}$. We can then define a function $f : [0,1] \to \mathfrak{c}$ by setting $f(x_\alpha) = \kappa_\alpha$, and mapping all remaining points of $[0,1]$ to (say) 0. For each $\kappa < \mathfrak{c}$ let $F_\kappa := f^{-1}(\{\kappa\})$. Now, for every null Borel set $N \in \mathcal{Z}$ and every $\kappa < \mathfrak{c}$, we have $F_\kappa \nsubseteq N$. Indeed, since $(N, \kappa) \in \mathcal{Z} \times \mathfrak{c}$, there is some α with $(N_\alpha, \kappa_\alpha) = (N, \kappa)$, which means that $x_\alpha \in F_\kappa$ and $x_\alpha \notin N$. Now for each $I \subseteq \mathfrak{c}$, define $A_I = \bigcup_{\kappa \in I} F_\kappa$. If $I \neq J$, there is some $\kappa' \in I \triangle J$. Then $F_{\kappa'} \subseteq A_I \triangle A_J$. If $A_I \triangle A_J$ were contained in some null Borel set N, then $F_{\kappa'} \subseteq N$, contradicting the earlier result. Therefore $A_I \triangle A_J$ is not contained in any null Borel set. But because λ is regular and complete, every λ-null measurable set is contained in a null Borel set; hence, $A_I \triangle A_J$ is not λ-null, which means that A_I and A_J are not λ-equivalent. Thus, the map $I \mapsto [A_I]_\lambda$ is injective from $\mathcal{P}(\mathfrak{c})$ into $[0,1]_\lambda$, and so $|[0,1]_\lambda| \geq 2^{\mathfrak{c}}$. $\square$